Agglomeration-based algebraic multigrid for linear systems coming from edge-element discretizations

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***Abstract*—An algebraic multigrid algorithm based on element agglomeration is proposed for linear systems coming from edge-element discretizations. The edge prolongation operator satisfies commutativity and energy-minimization properties. Robustness is illustrated on 2D numerical examples.**

Multigrid methods are beyond the most performing linear system solvers and these solvers are a key point in finite element method libraries. Here, we focus on the discretization on a triangular mesh with the lowest order edge element of

$$\operatorname{curl} \delta \operatorname{curl} \mathbf{U} + \gamma \mathbf{U} = \mathbf{f} \text{ on } \Omega \subset \mathbb{R}^2, \qquad (1)$$

which gives rise to a linear system $Ax = f$. As solver we propose an Algebraic MultiGrid (AMG) algorithm which keeps a good efficiency with non-smooth functions δ and γ and with unstructured triangulations. The coarsening process is based on *element agglomeration*. A *low-energy* edge prolongator satisfying a *commutativity property* [1] is obtained from the solution of local linear systems. Numerical examples demonstrate the robustness of our method.

I. Components of the algorithm

A. Levels

Starting from an initial fine grid partitioned into triangles, we build a hierarchy of generalized meshes $\tau^h = (\mathcal{D}^h, \mathcal{A}^h, \mathcal{S}^h)$. $\mathcal{D}^h = \left\{D_i^h\right\}_{i \in I_{\mathcal{D}^h}}$ is a family of open polygons we call elements. On the finest mesh they are the original triangles. $\mathcal{A}^h = \left\{A_i^h\right\}_{i \in I_{\mathcal{A}^h}}$ is a family of open one-dimensional manifolds we call edges. $\mathcal{S}^h = \left\{S_i^h\right\}_{i \in I_{\mathcal{S}^h}}$ is a family of single points we call vertices. The coarsening procedure which builds a generalized mesh $\tau^H = (\mathcal{D}^H, \mathcal{A}^H, \mathcal{S}^H)$ from a generalized mesh $\tau^h = (\mathcal{D}^h, \mathcal{A}^h, \mathcal{S}^h)$ must satisfy constraints defined in [2], [3]. Moreover, any edge in family $\mathcal{A}^H$ is a connected path of edges belonging to $\mathcal{A}^h$ and has two different endpoints which are vertices in $\mathcal{S}^H$ (Fig. 1). From the family of generalized mesh $\tau^h = (\mathcal{D}^h, \mathcal{A}^h, \mathcal{S}^h)$, we define a family of oriented simple graphs. For each vertex S_p^h of $\mathcal{A}^h$ a node p is created and 2 separate nodes p, q are connected by an arc each time an edge of $\mathcal{A}^h$ has S_p^h and S_q^h as endpoints. An arc starts from a node p to a node q if $p < q$. The arc-node incidence matrices denoted G^h can be viewed as discrete gradient operators on the meshes.

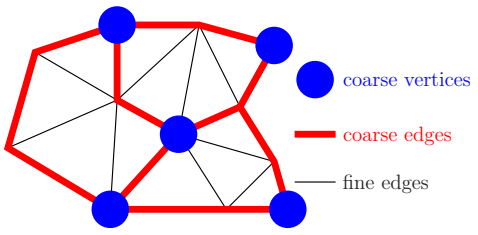


Fig. 1. Example of a coarse mesh τ^H builds from the finest mesh τ^h.

B. Prolongation operator properties

For each couple of generalized meshes (τ^h, τ^H), τ^H being obtained from τ^h by a coarsening procedure, we give us a nodal prolongator P^{nod} with a prescribed non-zero pattern and with all row sums equal to 1. We build then an edge prolongator P^{edg} with a prescribed non-zero pattern and with low-energy columns $(P^{\text{edg}})^e$, $e \in I_{\mathcal{A}^h}$. It must also satisfy a node-edge commutativity property [1]: $P^{\text{edg}} G^H = G^h P^{\text{nod}}$. The energy function on a mesh τ^h is defined by

$$g\left(x^e,\ e \in I_{\mathcal{A}^h}\right) = \sum_{e \in I_{\mathcal{A}^h}} \|x^e\|^2_{K^h} = \sum_{e \in I_{\mathcal{A}^h}} (x^e)^t K^h x^e \qquad (2)$$

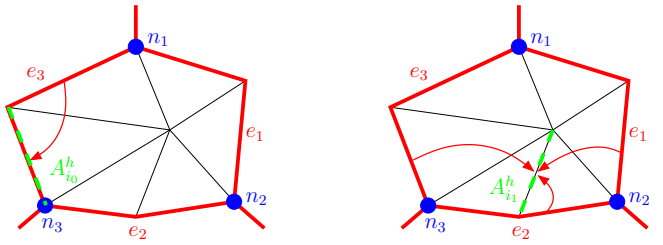
where K^h is a sparse symmetric definite positive matrix. For its non-zero pattern we require that $K^h_{i,i'} = 0$ if both edges A_i^h and $A_{i'}^h$ do not belong to the interior of the closure of the same element. Such a property is satisfied on the finest mesh by the finite element matrix A.

To define the non-zero pattern of the nodal prolongator P^{nod} and the edge prolongator P^{edg} we mimic the situation in geometric multigrid on nested meshes:

- If S_p^h belongs to the interior of the closure of an element of $\mathcal{D}^H$, $P^{\text{nod}}_{p,n} = 0$ if S_n^H does not belong to its boundary.
- If S_p^h is an endpoint of an edge of $\mathcal{A}^h$ included in edge A_e^H of $\mathcal{A}^H$ with endpoints $S_{n_k}^H$, $k = 1, 2$; if $S_p^h = S_{n_k}^H$, $P^{\text{nod}}_{p,n} = 0, \forall n \neq n_k$, else $P^{\text{nod}}_{p,n} = 0, \forall n \neq n_1, n_2$.
- If edge A_i^h of $\mathcal{A}^h$ is included in edge A_e^H of $\mathcal{A}^H$, $P^{\text{edg}}_{ee'} = 0, \forall e' \neq e$ (Fig. 2(a)).
- If edge A_i^h of $\mathcal{A}^h$ is included in the interior of the closure of an element of $\mathcal{A}^H$, $P^{\text{edg}}_{i,e} = 0$ if A_e^H is not included in its boundary (Fig. 2(b)).

C. Solution of the constrained minimization problem

An element D_a^H of $\mathcal{D}^H$ being fixed, we denote by $\left\{A_i^h\right\}_{i \in I_{\text{int}}}$ (resp. $\left\{A_i^h\right\}_{i \in I_{\text{bound}}}$) the family of edges of $\mathcal{A}^h$ included in the interior of its closure $\overline{D_a^H}$ (resp. included in



(a) $A^h_{i_0} \subset A^H_{e_3}$. (b) $A^h_{i_1} \in$ interior of $\overline{D^H_a}$.

Fig. 2. A coarse element D^H_a. Coarse edges are in wide lines ($m = 3$). Coarse vertices are large bullets. Arrows indicate prolongation.

its boundary), $A^H_{e_1}, A^H_{e_2}, \ldots, A^H_{e_m}$ the path of the edges of $\mathcal{A}^H$ included in its boundary. This path defines on the coarse graph an elementary cycle of nodes denoted $n_1, n_2, \ldots, n_m, n_1$.

Let A^h_i of $\mathcal{A}^h$ with starting point p and ending point q. Two different cases have to be considered :

1) If $i \in I_{\text{bound}}$, then A^h_i is included in an edge A^H_e of $\mathcal{A}^H$ (Fig. 2(a)). The non-zero entries of row i of P^{edg} are settled by the commutativity property since it writes
$$P^{\text{edg}}_{i,e} = P^*_{i,e} \text{ with } P^*_{i,e} = P^{\text{nod}}_{p,n} - P^{\text{nod}}_{q,n} \tag{3}$$
and S^H_n the starting endpoint of A^H_e.
2) If $i \in I_{\text{int}}$ (Fig. 2(b)) the commutativity property writes
$$\begin{aligned}\epsilon_{n_k}(e_{k-1})P^{\text{edg}}_{i,e_{k-1}} + \epsilon_{n_k}(e_k)P^{\text{edg}}_{i,e_k} \\ = P^{\text{nod}}_{p,n_k} - P^{\text{nod}}_{q,n_k}, \ \forall k \in [\![2; m]\!],\end{aligned} \tag{4}$$
where $\epsilon_{n_k}(e)$ denotes 1 (resp. -1) if n_k is the starting point (resp. ending point) of the arc e.
We deduce that the non-zero entries of row i of P^{edg} are defined up to a real constant θ_i as follows :
$$\begin{aligned}P^{\text{edg}}_{i,e_k} &= \epsilon_{n_k}(e_k)P^*_{i,e_k} + \theta_i\epsilon_{n_k}(e_k) \\ \text{with } P^*_{i,e_k} &= \left[\sum_{l=2}^{k} P^{\text{nod}}_{p,n_l} - \sum_{l=2}^{k} P^{\text{nod}}_{q,n_l}\right].\end{aligned} \tag{5}$$
It can be proved that the prolongator with the lowest energy is obtained with the values θ_i, $i \in I_{\text{int}}$ solution to a local linear system :
$$K^{D^H_a}\theta = K^{\overline{D^H_a}}b,$$
where $K^{D^H_a}$ (resp. $K^{\overline{D^H_a}}$) is the matrix obtained from the matrix K^h by keeping the entries $K^h_{i,j}$, $i, j \in I_{\text{int}}$ (resp. $K^h_{i,j}$, $i \in I_{\text{int}}, j \in I_{\text{int}} \cup I_{\text{bound}}$). b is given by:
$$b_i = -\frac{1}{m}\sum_{k=1}^{m} \epsilon_{n_k}(e_k)P^*_{i,e_k}, \ i \in I_{\text{int}} \cup I_{\text{bound}}.$$

II. Numerical results

The robustness of the method with parameters δ and γ is illustrated using examples of [4]. The domain is simply a unit square and Dirichlet boundary conditions are enforced.

In our strategy (Aggl. in the tables), coarse elements are obtained by double-pairwise agglomeration of elements. The results obtained by our method are compared with the Reitzinger and Schöberl (RS in the tables) strategy [1]. The comparison is based on the convergence rate σ_{est} in energy norm: $\sigma_{\text{est}} = (\text{er}^t_k A \text{er}_k / \text{er}^t_0 A \text{er}_0)^{1/(2k_f)}$ with er_k the error at the k-th iteration and k_f the iteration where the stopping criterion is reached. The smoother is a symmetric version of this proposed in [5] in a geometric multigrid context ((pre-, post-) smoothing steps in the tables).

For the homogeneous case in Table I, the two-grid convergence rate is quasi-independent of the size of the problem for both methods but the convergence rate of our method is better than for RS method and should lead to an optimal multigrid strategy. Moreover, increasing the number of pre- and post-smoothing steps significantly improves the convergence rate for the agglomeration. This behavior is confirmed in Table II and Table III for oscillating and discontinuous δ coefficients and two-grid and multigrid solver respectively. For this example, $f = C(2 + \sin(40\pi x))^2(2 + \cos(40\pi y))^2$ with $C = 10$ in $]0, 0.5[\times]0, 0.5[$, 10^4 in $]0.5, 1[\times]0, 0.5[$, 10^{-1} in $]0, 0.5[\times]0.5, 1[$ and 10^2 in $]0.5, 1[\times]0.5, 1[$.

TABLE I
RESULTS OBTAINED WITH A TWO-GRID SOLVER AND $\delta = \gamma = 1$.

	d.o.f. fine grid	736	3008	12160	48896
Aggl.	d.o.f. coarse grid	232	976	4000	16192
	(1, 1), σ_{est}	0.22	0.22	0.22	0.22
	(2, 2), σ_{est}	0.08	0.07	0.07	0.12
	(3, 3), σ_{est}	0.04	0.04	0.04	0.04
RS	d.o.f. coarse grid	184	751	3040	12224
	(1, 1), σ_{est}	0.62	0.68	0.70	0.71
	(3, 3), σ_{est}	0.52	0.64	0.69	0.70

TABLE II
RESULTS OBTAINED WITH A TWO-GRID SOLVER AND $\delta = f(x, y)$ AND $\gamma = f(y, x)$.

	d.o.f. fine grid	736	3008	12160	48896
Aggl.	d.o.f. coarse grid	232	976	4000	16192
	(1, 1), σ_{est}	0.39	0.41	0.32	0.21
RS	d.o.f. coarse grid	187	788	3087	12404
	(1, 1), σ_{est}	0.61	0.72	0.87	0.68

TABLE III
RESULTS OBTAINED WITH A W-CYCLE SOLVER AND $\delta = f(x, y)$ AND $f(y, x)$ ON THE GRID WITH 48896 DOFS.

	# grids	3	4	5	6	7
Aggl.	d.o.f. coarsest grid	4000	976	232	52	10
	(1, 1), σ_{est}	0.37	0.37	0.37	0.37	0.37
RS	d.o.f. coarsest grid	3279	844	216	55	11
	(1, 1), σ_{est}	0.83	0.89	0.93	0.95	0.97

The method is robust in 2D. The main difficulty for 3D problems is to find a fast and reliable agglomeration algorithm.

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